Lecture 7

6.8 - Indeterminate Forms and L'Hôpital's Rule

In Calc I, you computed the limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

using a complicated method. This was necessary because

$$\text{"} \frac{\sin 0}{0} = \frac{0}{0} \text{"} \ldots$$

This is called an <u>indeterminate form of the type $\frac{0}{0}$</u>

Other examples of this are:

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{-3}} \quad \text{and} \quad \lim_{x\to 2} \frac{\cos(\frac{\pi}{x})}{x-2}$$

An <u>indeterminate form of type $\frac{0}{0}$</u> is more formally the limit of a quotient $\frac{f(x)}{g(x)}$ where both $f(x), g(x) \to 0$.

Likewise, we can define an <u>indeterminate form of type $\frac{\infty}{\infty}$</u> as the limit of a quotient $\frac{f(x)}{g(x)}$ where $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$.

Examples of this type:

$$\lim_{x\to\infty} \frac{x^2}{e^x}, \quad \lim_{x\to -\frac{\pi}{2}^+} \frac{\tan x}{\ln\left(x+\frac{\pi}{2}\right)}, \quad \lim_{x\to 0^-} \frac{x^{-2}}{\csc x}$$

L'Hôpital's Rule

Let $\lim$ stand for any of

$$\lim_{x\to a},\ \lim_{x\to a^+},\ \lim_{x\to a^-},\ \lim_{x\to\infty},\ \text{or}\ \lim_{x\to-\infty}$$

and suppose $\lim \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then, if

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Ex: $\lim_{x\to 0} \frac{\sin x}{x}$

$\lim f(x)g(x)$ is an indeterminant form of type $0 \cdot \infty$
if $\lim f(x) = 0$ & $\lim g(x) = \infty$

An example of this is $\lim\limits_{x\to\infty} x^3 e^{-x^2}$

The method of dealing with this type is to turn it into either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. To turn it into:

$\frac{0}{0}$: write $f(x)g(x) =$

$\frac{\infty}{\infty}$: write $f(x)g(x) =$

Ex: $\lim\limits_{x\to\infty} x^3 e^{-x^2}$

Other common types of indeterminant forms are of the type 0^0, ∞^0, 1^∞: These involve limits of the form $\lim [f(x)]^{g(x)}$ where:

0^0:	∞^0:	1^∞:
$f(x) \to 0$	$f(x) \to \infty$	$f(x) \to 1$
$g(x) \to 0$	$g(x) \to 0$	$g(x) \to \infty$

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The method to deal with these is the same in each case:

Since e^x is continuous, and $f(x)^{g(x)} = e^{g(x)\ln[f(x)]}$,

to compute $\lim f(x)^{g(x)}$:

1) Compute

2) Call this limit α. α may be finite, or $\pm\infty$

3) Then $\lim f(x)^{g(x)} = \lim\limits_{t \to \alpha} e^t$

<u>Ex</u>: $\lim\limits_{x \to 0^+} x^{\sqrt{x}}$

The last indeterminant form is of <u>type $\infty - \infty$</u>:
this occurs with limits of the form $\lim (f(x) - g(x))$
where $f(x), g(x) \to \infty$ or $f(x), g(x) \to -\infty$.

The way to deal with these are to try to turn them into quotients, and then apply the previous methods.

<u>Ex</u>: $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

<u>Ex</u>: $\lim_{x \to 0} (\csc x - \cot x)$